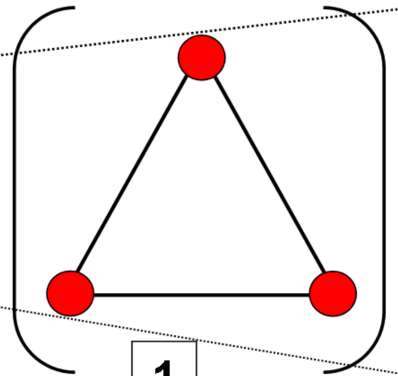
THEOREM OF THE DAY

The Bruck-Ryser-Chowla Theorem *If a projective plane of order* n *exists, with* $n \equiv 1$ *or* $2 \pmod 4$ *then* $n = x^2 + y^2$ *for some integers* x *and* y*.*



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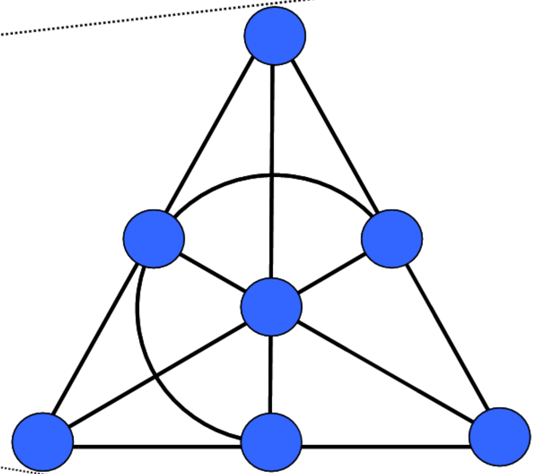

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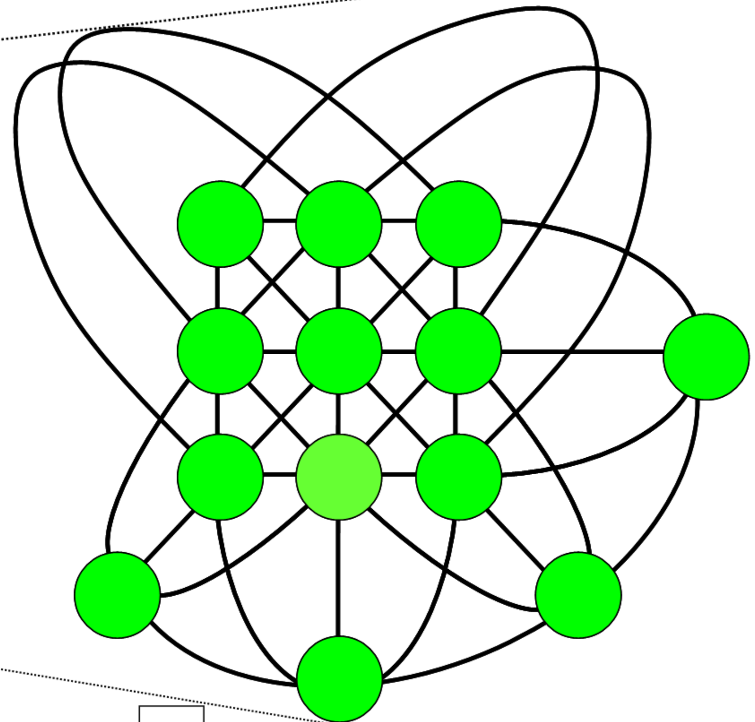

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A finite projective plane of order n has $n^2 + n + 1$ points and $n^2 + n + 1$ lines, arranged so that

1. every line contains $n + 1$ points, and every point is on $n + 1$ lines,
2. any two distinct lines intersect at exactly one point, and any two distinct points lie on exactly one line.

The first three planes are shown above. An extra condition—there is some set of four points no three of which are collinear—is often added to exclude degenerate possibilities and this eliminates the plane of order 1.

One of the most famous problems in combinatorics is to confirm or disprove that finite projective planes only exist for prime power order. The Bruck-Ryser-Chowla theorem (proved by Richard Bruck and Herbert Ryser in 1949 and generalised a year later by Ryser and Sarvadaman Chowla) eliminates many possible counterexamples, starting with $6 = 4 + 2$, $14 = 3 \times 4 + 2$, $21 = 5 \times 4 + 1$ and $22 = 5 \times 4 + 2$, which are not sums of squares. The numbers $10 = 1^2 + 3^2$, $12 \equiv 0 \pmod 4$, $15 \equiv 3 \pmod 4$ and $18 = 3^2 + 3^2, \ldots$ escape the net. In 1989 Clement Lam, John McCay, Stanley Swiercz and Larry Thiel, building on work of Larry Carter in the 1970s, resolved the first of these cases, using a Herculean combination of mathematical reasoning and computer search: **there is no projective plane of order 10**. The cases 12, 15 and 18 do not seem to be within reach at present.

Web link: the www.maa.org/programs/maa-awards/writing-awards/the-search-for-a-finite-projective-plane-of-order-10

Further reading: *Designs, Graphs, Codes and their Links* by P. J. Cameron and J. H. van Lint, Cambridge University Press, 1991.