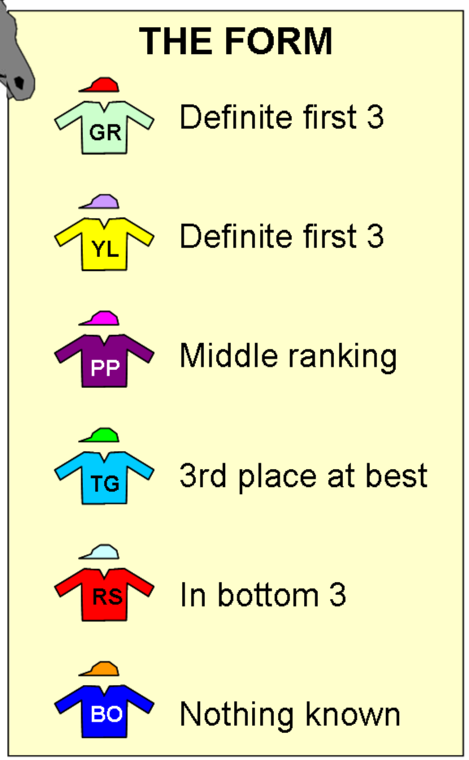
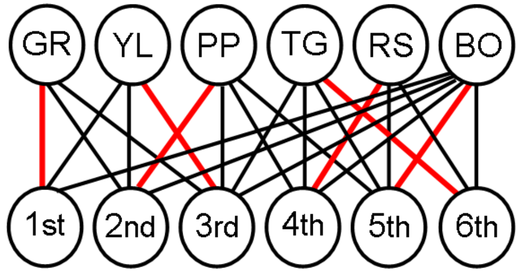
THEOREM OF THE DAY

Bregman's Theorem *Let A be an* $n \times n$ $(0,1)$*-matrix, with non-zero row sums* r_i, $i = 1, \ldots, n$. *Then the permanent of A satisfies the inequality,*

$$per(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$





	1st	2nd	3rd	4th	5th	6th
GR	(1)	1	1	0	0	0
YL	1	1	(1)	0	0	0
PP	0	(1)	1	1	1	0
TG	0	0	1	1	1	(1)
RS	0	0	0	(1)	1	1
BO	1	1	1	1	(1)	1

What are the odds of predicting the correct order of every horse in a six-horse race? Complete ignorance of racing is equivalent to saying any of the $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$ orderings is equally likely, and any choice has a probability of $1/6! = 1/720 \approx 0.0014$ of success. Racing experts, however, are able to do much better. Their estimates (e.g. above, centre) of the horses' chances may be combined in a bipartite graph (above, top-right) in which the possible *matchings* of horses to final positions are considerably restricted. If a (0,1)-matrix is used to represent the graph, as shown on the far left, then its *permanent* enumerates precisely the number of ways of simultaneously choosing a 1 from each row and column. The red, circled, 1's show one way of doing this; there are 56 in total, this being the value of the permanent of this matrix. The chance of correct prediction has increased to $1/56 \approx 0.018$. For large matrices, no known method will yield the value of the permanent function in a practical amount of time. Bregman's inequality gives a good upper bound, with value almost exactly 88 for our example; equality is obtained precisely when the matrix consists, up to row and column permutations, of diagonal blocks of all-1's.

This celebrated inequality was conjectured by Henryk Minc in 1963 and proved by Lev Bregman ten years later.

Web link: www.ams.org/journals/bull/1979-01-06/home.html; click on Richard A. Brualdi's review of Minc's *Permanents*. Proof and extensions of Bregman: www3.nd.edu/~dgalvin1/pdf/bregman.pdf.

Further reading: *The Probabilistic Method, 3rd ed.* by Noga Alon and Joel H. Spencer, WileyBlackwell, 2008.