

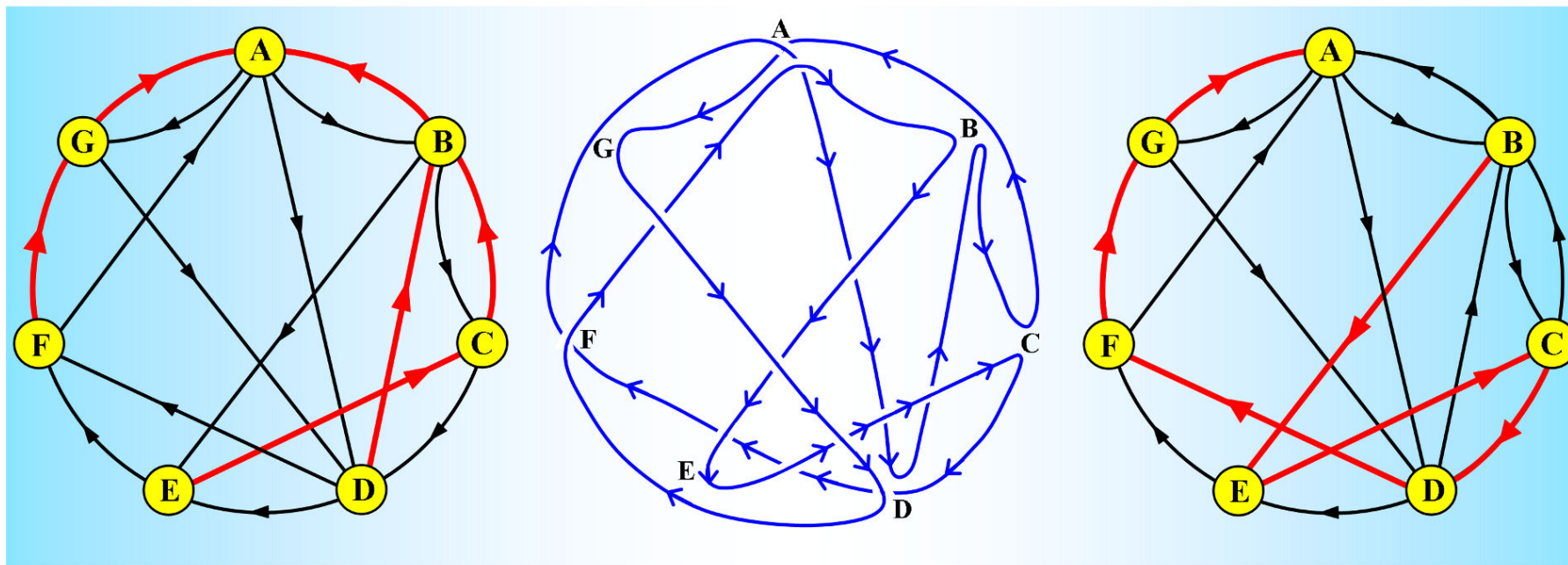


THEOREM OF THE DAY

The BEST Theorem Let $G = (V, E)$ be a directed graph in which, for each vertex v in V , the indegree and outdegree have the same value, $d(v)$, say. Then G has a directed Euler tour: a closed walk which passes each edge exactly once; let $\varepsilon(G)$ denote the number of such tours. Then, for any fixed vertex x ,

$$\varepsilon(G) = t_x \prod_{v \in V} (d(v) - 1)!$$

where t_x denotes the number of those spanning trees of G in which every vertex has a directed path to x .



A directed graph is shown here with (on the left and right) two spanning trees directed towards vertex A . Choose an edge from A , and from each vertex successively reached, with the rule that tree edges are the last exit chosen from any vertex. The exits from vertex v may be chosen in $(d(v) - 1)!$ orderings; in each case a different Euler tour results. Starting with edge $A \rightarrow G$ and using the left-hand tree, the tour shown above centre is one outcome. This tree allows tours to order their exits from A in $d(A)! = 6$ ways but each resulting tour, if traversed commencing with a different initial edge from A , will be found to be counted again by some different tree. Thus, if we follow the tour in the centre starting along edge AD , we will find that the final exit edges are chosen from the right-hand tree. Starting with edge AB will identify a third tree. Thus vertex A contributes a factor of $d(A)!/d(A) = (d(A) - 1)!$ to the product formula, so that it contributes in just the same way as the other vertices albeit for a different reason. By the way, the centre image is reminiscent of a knot diagram and there is indeed a fascinating connection to explore!

The special case of this theorem in which $d(v) = 2$ for every vertex was proved in 1941 by Cedric Smith and Bill Tutte. Nicolaas de Bruijn and Tatyana van Aardenne-Ehrenfest provided the first two initials for the theorem's nickname in 1951, proving the general case and making explicit the link to spanning trees.

Web link: www.cdam.lse.ac.uk/Reports/Files/cdam-2004-12.pdf; see concretenonsense.wordpress.com/2009/08/20/) for a nice account of the connection to knot theory.

Further reading: *A Course in Enumeration* by Martin Aigner, Springer, 2007, Chapter 9.

