THEOREM OF THE DAY

Stirling's Approximation *For positive integers n, the value of the factorial function n! is given asymptotically by*

$$n! \approx \sqrt{\tau n}\, n^n e^{-n},$$

where $\tau = 2\pi$.

Pascal's triangle $n = 0 \ldots 12$.	Using Stirling's approximation (& rounding)	Ratio of actual to unrounded approximation
1	1	1
1, 1	1, 1	1, 1
1, 2, 1	1, 2, 1	1, 0.885, 1
1, 3, 3, 1	1, 3, 3, 1	1, 0.909, 0.909
1, 4, 6, 4, 1	1, 4, 6, 4, 1	1, 0.917, 0.940, 0.
5, 10, 10, 5, 1	1, 5, 11, 11, 5, 1	1, 0.916, 0.943, 0.943
15, 20, 15, 6, 1	1, 7, 16, 21, 16, 7, 1	1, 0.916, 0.955, 0.962, 0.
1, 35, 35, 21, 7, 1	1, 8, 22, 36, 36, 22, 8, 1	1, 0.920, 0.955, 0.964, 0.964
56, 70, 56, 28, 8, 1	1, 9, 29, 58, 72, 58, 29, 9, 1	1, 0.916, 0.956, 0.966, 0.968, 0.
, 126, 126, 84, 36, 9, 1	1, 10, 38, 87, 130, 130, 87, 38, 10, 1	1, 0.921, 0.952, 0.966, 0.969, 0.969
10, 252, 210, 120, 45, 10, 1	1, 11, 47, 124, 216, 258, 216, 124, 47, 11, 1	1, 0.917, 0.949, 0.976, 0.972, 0.969, 0.
, 462, 462, 330, 165, 55, 11, 1	1, 12, 57, 170, 338, 473, 473, 338, 170, 57, 12, 1	1, 0.924, 0.953, 0.971, 0.973, 0.981, 0.981
92, 924, 792, 495, 220, 66, 12, 1	1, 13, 69, 227, 507, 809, 943, 809, 507, 227, 69, 13, 1	1, 0.923, 0.955, 0.973, 0.976, 0.977, 0.980, 0.

Since binomial coefficients $\binom{n}{k}$ can be calculated as $\binom{n}{k} = n!/k!(n-k)!$ we can use Stirling's approximation to calculate the approximate values in Pascal's triangle, as shown here. The boxed values locate the so-called **central binomial coefficients**, for which our approximation simplifies neatly to $\binom{n}{n/2} \sim 2^{n+1}/\sqrt{\tau n}$. They indicate that Stirling's approximation is increasing in accuracy as n increases. The distance between the actual and approximated values grows larger with n but their ratio grows closer and closer to one so that, *asymptotically,* the approximation is accurate. The approximation may also be stated as an equality $n! = \sqrt{\tau n}(n/e)^n(1 + O(1/n))$. The $O(1/n)$ represents additional terms which vanish as quickly as or quicker than $1/n$; it may be 'expanded' as far as desired: Knuth replaces $1 + O(1/n)$ with $1 + 1/(12n) + 1/(288n^2) - 139/(51840n^3) - 571/(2488320n^4) + O(1/n^5)$ which, rounding to the nearest integer, gives the exact value of $n!$ up to $n = 12$.

James Stirling published his approximation in 1730. It would seem more fair to call it the De Moivre–Stirling approximation since Abraham de Moivre had just discovered that $n! \approx c\sqrt{n}\, n^n e^{-n}$, for some constant c. Stirling's contribution, acknowledged by De Moivre, was to identify the constant as $\sqrt{\tau}$. The formula thereby becomes one of those rare mathematical beasts (like Euler's $e^{\tau i/2} + 1 = 0$) that display in a natural way the fundamental constants e and τ.

Web link: www.stat.ualberta.ca/people/schmu/preprints/factorial.pdf

Further reading: *The Art of Computer Programming* by Donald E. Knuth, Addison-Wesley, 1999 edition. Vol. 1, section 1.2.11.