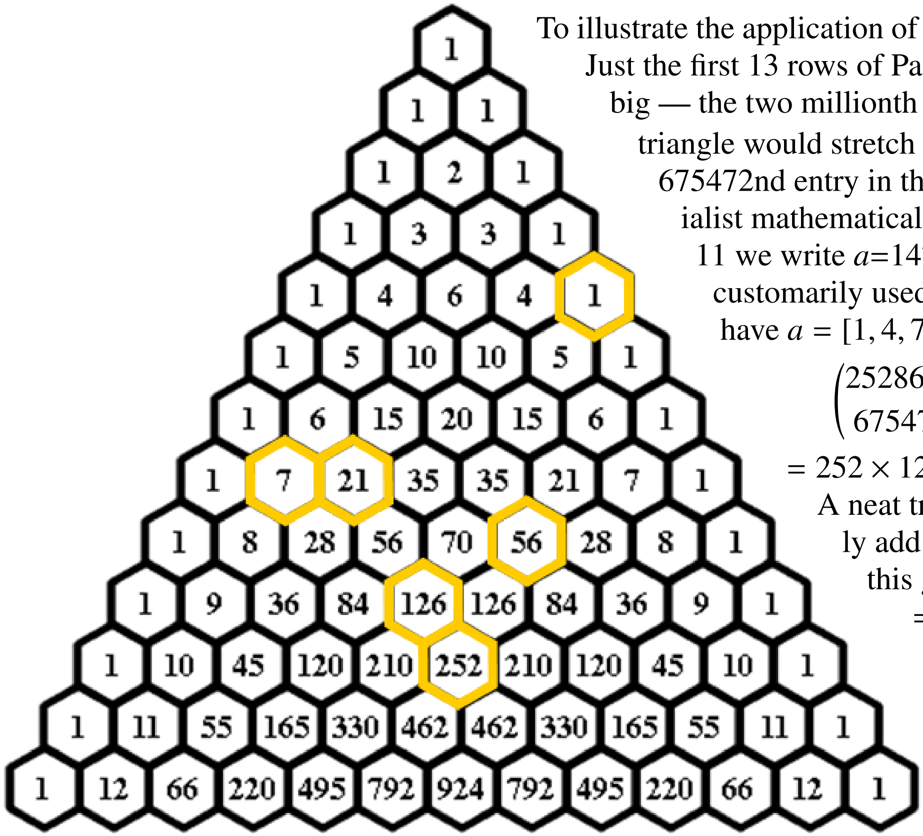
THEOREM OF THE DAY

Lucas' Theorem *Let p be a prime number and let a and b, $a \geq b$, be positive integers written in base p (say, $a = [a_s, a_{s-1}, \ldots, a_1, a_0]$, $b = [b_t, b_{t-1}, \ldots, b_1, b_0]$, with $a = \sum a_i p^i$ and $b = \sum b_j p^j$ and $s \geq t$). Then*

$$\binom{a}{b} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_{t-1}}{b_{t-1}}\binom{a_t}{b_t} \pmod{p}.$$



To illustrate the application of Lucas' theorem, we may consider the values $a = 2528646$ and $b = 675471$. Just the first 13 rows of Pascal's triangle are shown here (starting at row 0). The numbers get very big — the two millionth row contains almost a trillion digits (by which time, on this scale, the triangle would stretch beyond the moon!) So to calculate the precise value of $\binom{2528646}{675471}$, the 675472nd entry in the 2528647th row, is a big task, even for a computer running specialist mathematical software. But we can easily find its remainder, mod 11: in base 11 we write a=147789A$_{11}$ (in base conversions ≥ 10, the letters A, B, C, ..., are customarily used to represent 10, 11, 12, ...). In base 11, b=421545$_{11}$. So we have $a = [1, 4, 7, 7, 8, 9, 10]$ and $b = [4, 2, 1, 5, 4, 5]$ and, by Lucas' Theorem,

$$\binom{2528646}{675471} \equiv \binom{10}{5} \times \binom{9}{4} \times \binom{8}{5} \times \binom{7}{1} \times \binom{7}{2} \times \binom{4}{4} \pmod{11}$$

$= 252 \times 126 \times 56 \times 7 \times 21 \times 1 \pmod{11} = 261382464 \pmod{11}$. A neat trick for extracting the remainder mod 11 is to alternately add and subtract digits in reverse order; for 261382464 this gives $4 - 6 + 4 - 2 + 8 - 3 + 1 - 6 + 2 = 19 - 17 = 2$. And, in a mere few minutes, we have discovered something about a number having about 640 thousand digits: and that is more digits than there are minutes in sixty three whole weeks!

Edouard Lucas (1842–1891) is remembered for his work on the Fibonacci numbers and for inventing the Towers of Hanoi puzzle. His theorem on the binomial coefficients can be used to derive many interesting properties of Pascal's triangle, for instance, $\binom{a}{b}$ is odd exactly when every 1 in the base 2 representation of b is also a 1 in the base 2 representation of a.

Web link: www.math.hmc.edu/funfacts/ffiles/30002.4-5.shtml; the Pascal's triangle image of is adapted from one at mathforum.org.

Further reading: *Combinatorics: Topics, Techniques, Algorithms,* by Peter J. Cameron, CUP, 1994, chapter 3.