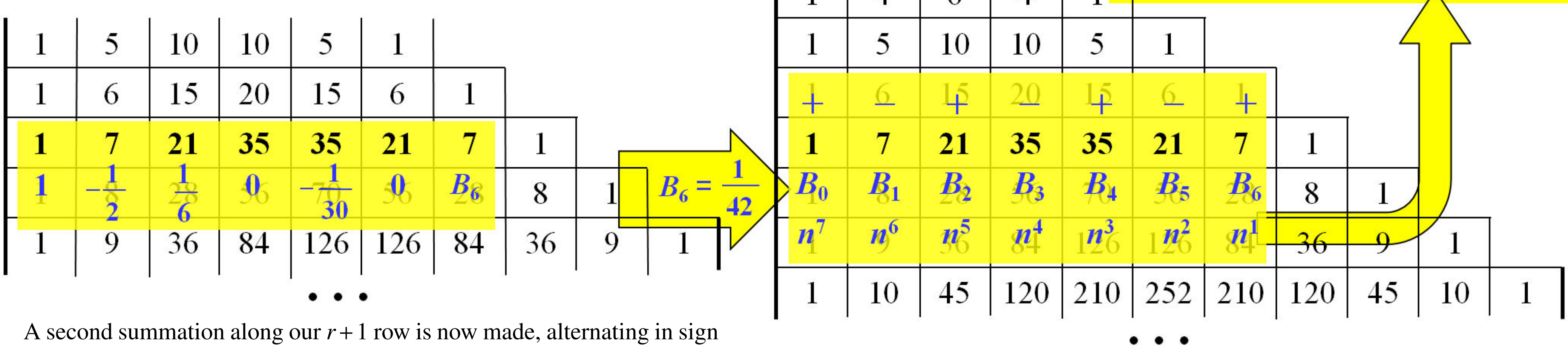
THEOREM OF THE DAY

Faulhaber's Formula *The sum of the r-th powers of the first n positive integers is given by*

$$1^r + 2^r + \ldots + n^r = \frac{1}{r+1}\sum_{k=0}^{r}(-1)^k\binom{r+1}{k}B_k n^{r-k+1}.$$

The calculation of our sum of r-th powers involves a double scan of the $(r+1)$-th row of Pascal's triangle. We need to produce the first $r+1$ so-called **Bernoulli numbers**, denoted by $B_0, B_1, \ldots, B_r$. Suppose that we have $B_0, B_1, \ldots, B_{r-1}$, then we can extract B_r by solving the equation $\sum_{i=0}^{r}\binom{r+1}{i}B_i = 0$. In the example below, $r = 6$; the values of $B_0, \ldots, B_5$ are shown ($B_0 = 1, B_1 = -1/2$, etc) and the equation yields the value $B_6 = 1/42$. (The properties of Pascal's triangle conspire to give every Bernoulli number of odd index, beyond B_1, the value zero.)

$$1^6 + 2^6 + 3^6 + 4^6 + 5^6 + \ldots + n^6$$
$$= \frac{1}{7}\left(n^7 + \frac{7}{2}n^6 + \frac{21}{6}n^5 - \frac{35}{30}n^3 + \frac{7}{42}n\right)$$
$$= \frac{1}{42}\left(6n^7 + 21n^6 + 21n^5 - 7n^3 + n\right)$$
$$= \frac{1}{42}n(n+1)(2n+1)\left(3n^4 + 6n^3 - 3n + 1\right)$$



A second summation along our $r+1$ row is now made, alternating in sign and multiplying each entry by Bernoulli numbers of ascending index and by descending powers of n. The result is a polynomial in n of degree $r+1$. For $r = 1$ the result is the familiar formula for the first n positive integers: $\frac{1}{2}n(n+1)$; the $1/2$ can be explained by simple combinatorial arguments but its appearance as the value of $-B_1$ cuts much deeper.

The study of sums of powers goes back to Greek times and was a preoccupation of medieval scholars in India and the Islamic world. Johann Faulhaber was the first, in 1631, to publish a systematic list of polynomials resembling that given above. The general formula, using Bernoulli numbers, is due to Jacob Bernoulli (1713) and, in what is surely one of the most remarkable instances of simultaneous discovery in mathematics, the Japanese 'Arithmetical Sage' Takakazu Seki (1712).

Weblink: www.maa.org/publications/periodicals/convergence/sums-of-powers-of-positive-integers.
Further reading: The Book of Numbers by John H. Conway, Copernicus, 1996.