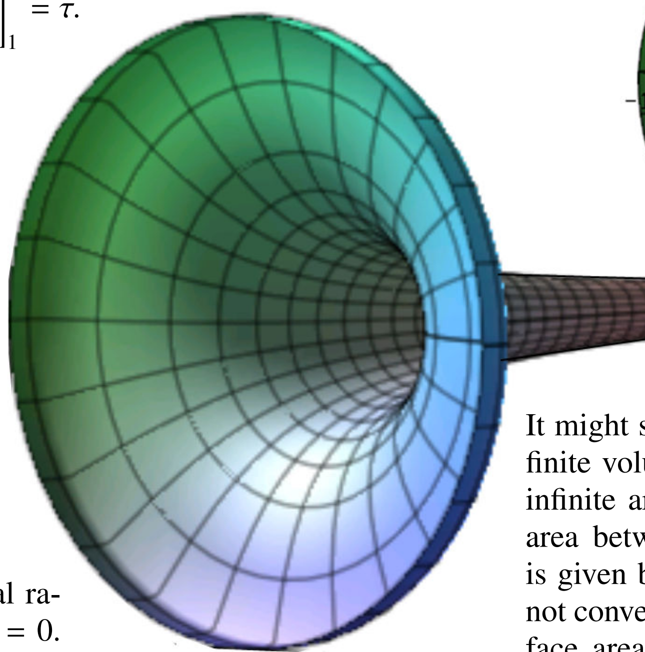
THEOREM OF THE DAY

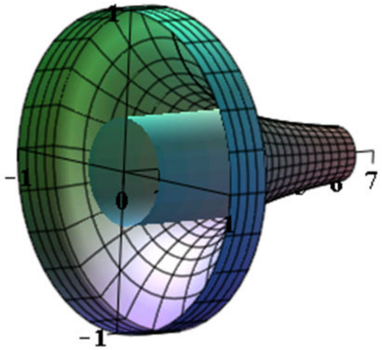
Torricelli's Trumpet *Suppose that the hyperbola $y = 1/x$ is rotated about the x axis, in the interval $[1, \infty)$, through one complete turn (τ radians) to form a solid, and to this is added a cylinder of radius and height 1. Then the combined volume is equal to that of a cylinder of height 1 and radius $\sqrt{2}$.*

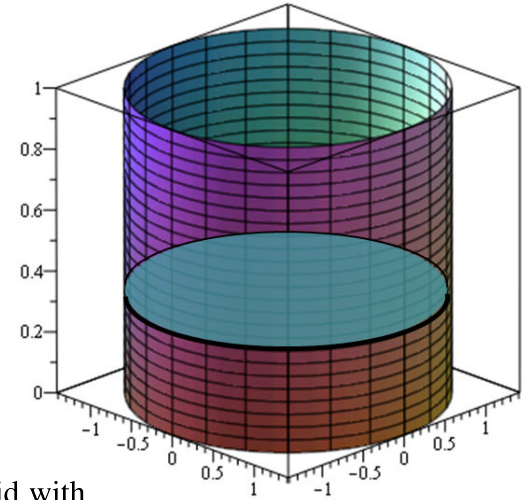
The cylinder+rotated hyperbola volume is calculated easily using integration:

$$\text{vol} = \frac{1}{2}\tau \times 1^2 + \frac{\tau}{2}\int_1^\infty \left(\frac{1}{x}\right)^2 dx = \frac{1}{2}\tau + \frac{\tau}{2}\left[-\frac{1}{x}\right]_1^\infty = \tau.$$

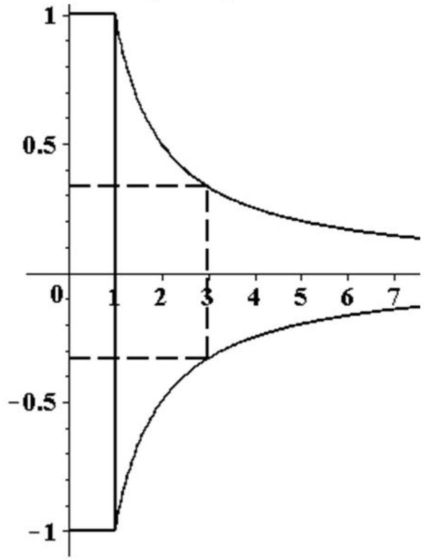
But the volume of this solid was first calculated before either Newton or Leibniz was born, using the 'method of indivisibles'. Thus we observe that any cylinder of radius r, placed inside the solid and stretching from the y axis to the hyperbola, will touch the hyperbola at the points $(1/r, r)$ and $(1/r, -r)$, and will have surface area $\tau r \times 1/r = \tau$ (the solid itself starts with the cylinder of radius $r = 1$). This surface area is identical to the area of a disk of radius $\sqrt{2}$. The illustration on the right shows the case $r = 1/3$ (top centre and bottom right), with the disk situated at height $1/3$ in a cylinder of radius $\sqrt{2}$ and height 1, as shown top right. Now, starting at $r = 1$, we reduce the cylindrical radius r, by infinitesimal amounts, from $r = 1$ to $r = 0$. The infinity of cylindrical surface areas thus formed exactly make up the combined volume of the initial cylinder and the rotated hyperbola. But they are equal in area to an infinity of disks of radius $\sqrt{2}$, located at infinitesimal intervals down a cylinder of height 1. So the two volumes are identical.







It might seem contradictory that a solid with finite volume should have a cross section of infinite area, but such is the case, since the area between the hyperbola and the x axis is given by $\int_1^\infty 1/x\,dx = [\ln x]_1^\infty$ which does not converge. Still more surprisingly, the surface area of the rotated hyperbola is given by $\tau \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx < \tau \int_1^\infty \frac{1}{x} \times 1\,dx$, again failing to converge: we have a finite volume enclosed by an infinite 'skin'! These apparent paradoxes are resolved when mathematical analysis is applied to the abstract constructs that are the real numbers and functions defined thereon but remain puzzling when imagined in the concrete world.



The method of indivisibles, usually known as **Cavalieri's Principle**, makes a correspondence between plane sections of a known and those of an unknown volume. Evangelista Torricelli (1608-1647) boldly and ingeniously extended this to non-plane surfaces (to remove any doubts he provided a second, classical, Euclid-style proof). Similar calculations with *plane* infinite figures had been made earlier, including by Nicolas Oresme in the 14th century, but Torricelli's 1644 publication, emanating from the circle of Galileo, made him known throughout Europe and caused a philosophical outcry.

Web link: fredrickey.info/hm/CalcNotes/default.htm.

Further reading: *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century* by Paolo Mancosu, Oxford University Press, 1999, chapter 5.