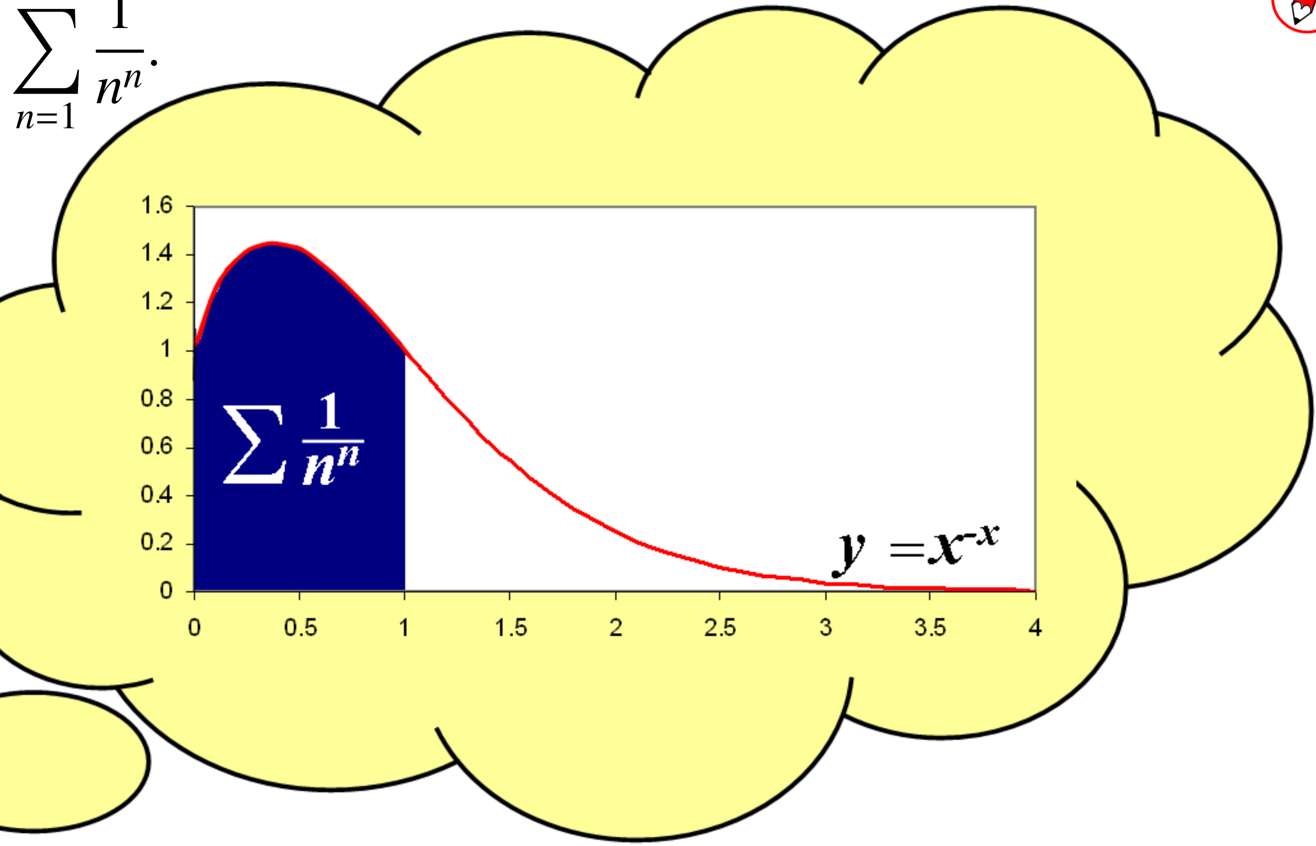
THEOREM OF THE DAY

The Sophomore's Dream

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

The indefinite integral $\int x^{-x}dx$ does not have a closed-form solution—it may be evaluated as a double summation of the form $\sum_{n=0}^{\infty}\sum_{i=0}^{n} f(n,i)x^{n+1}(\ln x)^{n-i}$, with f being a function of n and i. It so happens, when the integral is evaluated over the interval $[0,1]$, that $0^{n+1} = 0$ and $\ln(1) = 0$ conspire to collapse this double summation to the beautifully neat expression in this theorem. The infinite sum converges rapidly to the value of this definite integral, that is to the area beneath the curve $y = x^{-x}$ from $x = 0$ to $x = 1$, an accuracy of 20 decimal places (viz. $\int_0^1 x^{-x}dx \approx 1.29128599706266354041$) being achieved within 16 terms.

There is an analogous (but alternating) summation giving the value of $\int_0^1 x^x dx$.



This identity is due (1697) to the Swiss mathematician Johann Bernoulli, who taught Leonhard Euler and was a pioneer in finding applications for the calculus, developed in the 60s and 70s by Newton and Leibniz. It has become known as 'the sophomore's dream' in analogy with the 'freshman's dream' identity $(x+y)^n = x^n + y^n$ which, alas for the freshman, is false in general when x and y are real numbers (although true in fields of characteristic dividing n).

Web link: klotza.blogspot.fr/2015/11/the-sophomores-spindle-all-about.html

Further reading: *Experimentation in Mathematics: Computational Paths to Discovery* by Jonathan M. Borwein, David H. Bailey and Roland Girgensohn, A K Peters, 2008, section 1.1.