

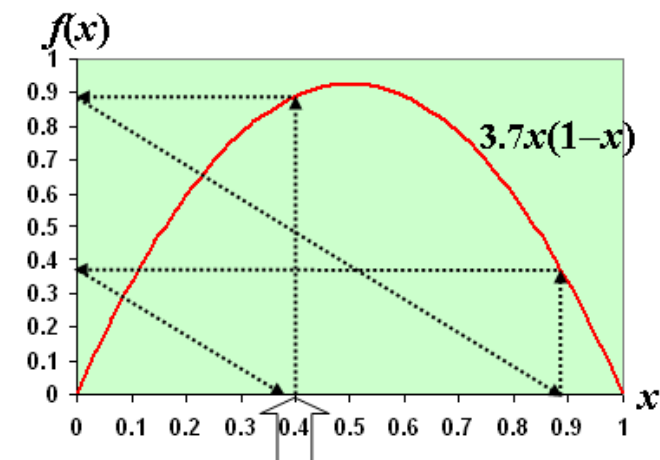


# THEOREM OF THE DAY

**Sharkovsky's Theorem** Specify an ordering,  $<$ , of the positive integers:

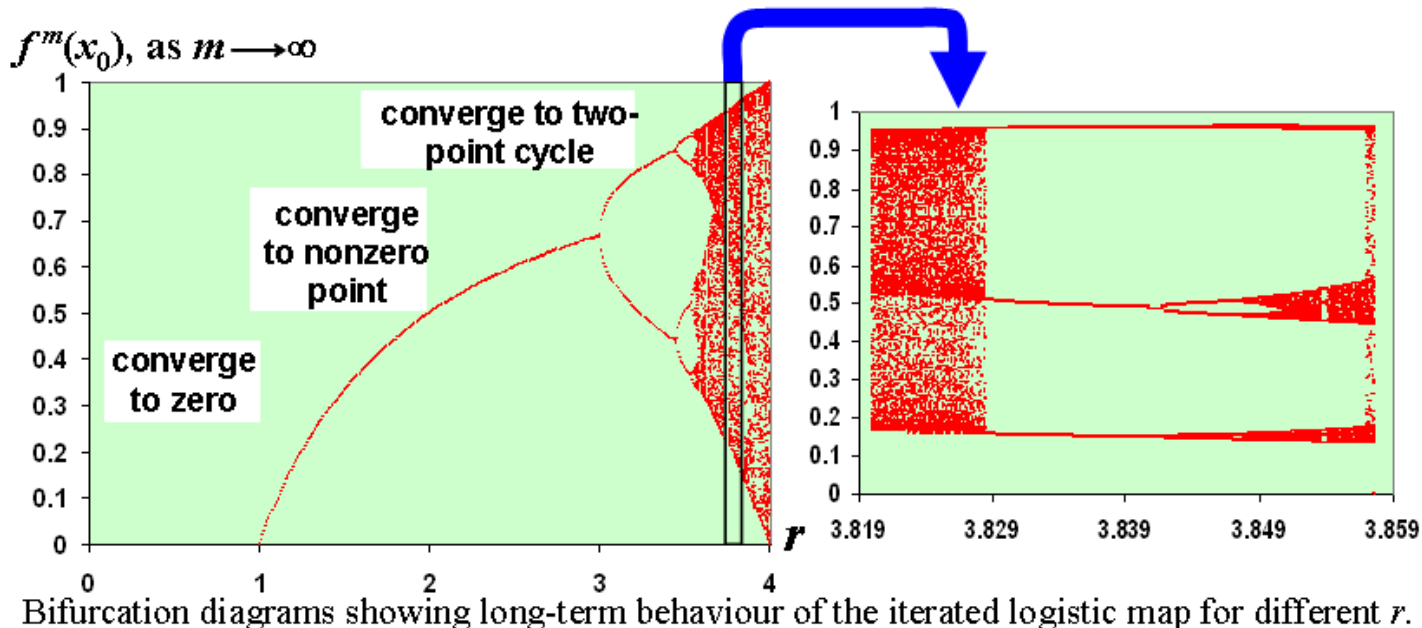
$$3, 5, 7, 9, \dots, 2 \times 3, 2 \times 5, 2 \times 7, 2 \times 9, \dots, 2^2 \times 3, 2^2 \times 5, 2^2 \times 7, 2^2 \times 9, \dots, 2^4, 2^3, 2^2, 2^1, 1,$$

defined formally as follows: take  $x < y$  with  $x$  and  $y$  written (uniquely) as  $x = 2^r p$  and  $y = 2^s q$ ,  $p, q$  odd; then  $x < y$  if  $r \leq s$  and  $p > 1$ ; otherwise  $y < x$ . Now let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function having a point  $x$  of period  $m$ ; that is,  $f^m(x) = f(x)$ , where  $f^m$  denotes the  $m$ -th iteration of  $f$ . Then for every  $n$  with  $m < n$ ,  $f$  has some point of period  $n$ . In particular, if  $f$  has a point of period 3, then  $f$  has periods of all positive integer orders.



Iterating the logistic map  $f(x) = rx(1-x)$  from  $x_0 = 0.4$ , using  $r = 3.7$ :

$$0.4 \rightarrow 0.89 \rightarrow 0.36 \rightarrow 0.85 \rightarrow \dots$$



For any choice of parameter  $r$ , the *logistic map*,  $f(x) = rx(1-x)$ , certainly gives a continuous curve, as is illustrated in the left-hand plot (using  $r = 3.7$ ). The long-term behaviour of the iterated logistic map is displayed in the right-hand plots: choose an arbitrary initial value  $x_0 \in (0, 1)$  and calculate the sequence  $f(x_0), f(f(x_0)), f(f(f(x_0))), \dots$ ; for values of  $r$  between zero and  $r \approx 3.57$  the sequence leads our arbitrary initial value to convergence in a unique cycle through  $2^i$  values,  $i = 0, 1, 2, \dots$  (period  $2^i$ ). Beyond 3.57, chaos ensues: indeed, the choice  $r = 3.7$  appears to provide no convergent behaviour, and a tiny change to  $x_0$  eventually causes a large change to our sequence. Suddenly, around  $r = 3.83$ , a window of order opens! On the far right, magnification shows convergence to a cycle of period three. But here Sharkovsky's theorem predicts periods of *any* length. Thus some choice of  $x_0$  will converge to a cycle of length, say, 100; This cycle will not be an *attractor*: an infinitesimal change to  $x_0$  will drive us back to period 3. Such non-attracting periods are invisible to computers!

The Ukrainian O.M. Sharkovsky's 1964 theorem shows that another world exists behind the complexity of fractal plots.

**Web link:** [www.mcasco.com/Order/ordintro.html](http://www.mcasco.com/Order/ordintro.html): an excellent free on-line course by J.D. Jones for M. Casco Associates.

**Further reading:** *Over and Over Again* by Gengzhe Chang and Thomas W. Sederberg, Mathematical Association of America, 1998, chapter 23.

