



true for low degree  
( $n \leq 8$ ), & high degree

# THEOREM OF THE DAY

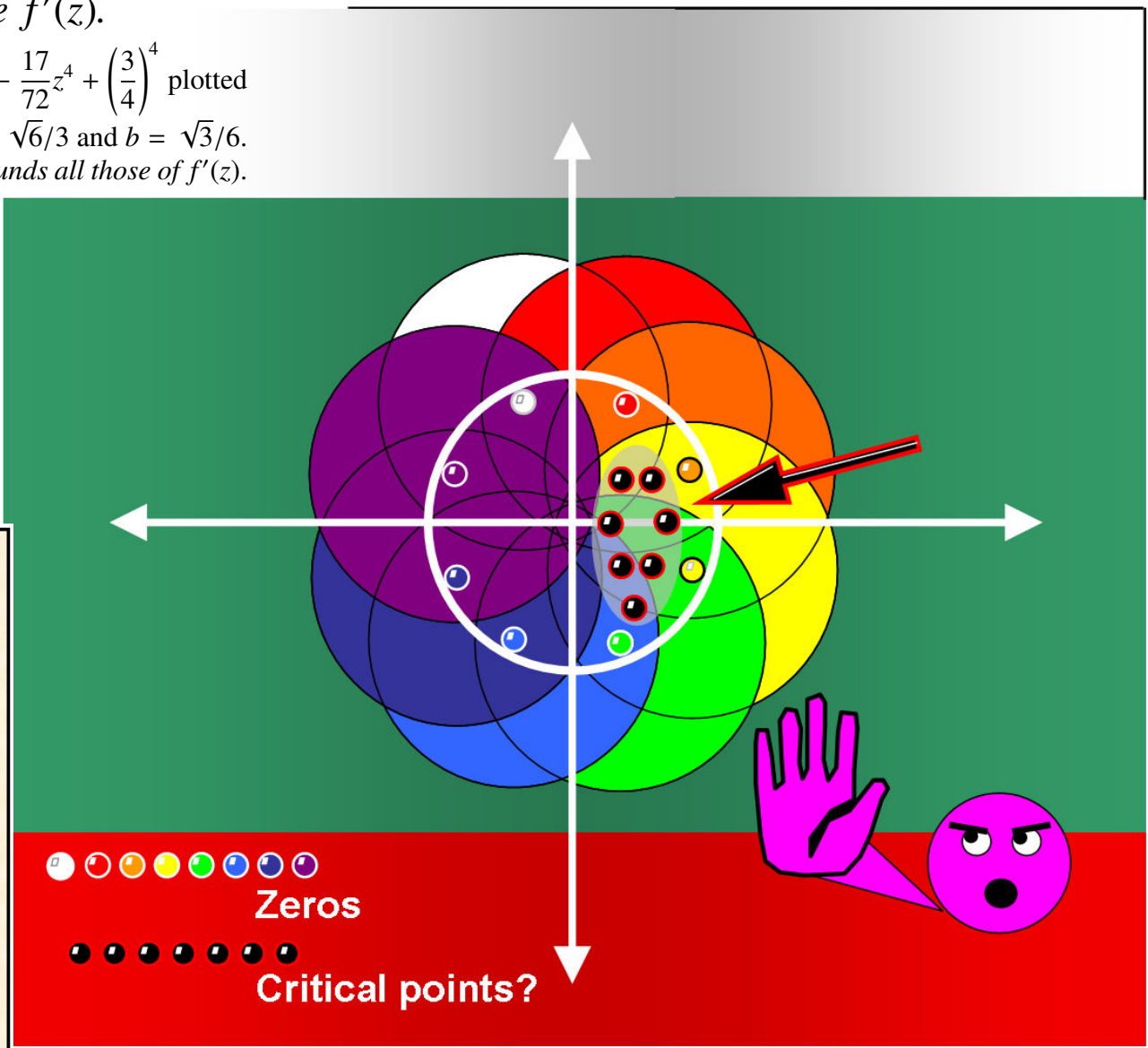


**Sendov's Conjecture (a Theorem Under Construction!)** Let  $f(z)$  be a polynomial of degree  $n \geq 2$ , all of whose zeros lie in the closed unit disk. Then for any zero  $z_0$  of  $f(z)$ , the closed unit disk with centre  $z_0$  contains at least one zero of the derivative  $f'(z)$ .

The sketch on the right shows the zeros of the polynomial  $f(z) = z^8 - \frac{17}{72}z^4 + \left(\frac{3}{4}\right)^4$  plotted

on the complex plane: these zeros are  $\pm a \pm bi$  and  $\pm b \pm ai$ , where  $a = \sqrt{6}/3$  and  $b = \sqrt{3}/6$ . By the **Gauss-Lucas Theorem**: the convex hull of the zeros of  $f(z)$  bounds all those of  $f'(z)$ .

So  $f'(z)$  certainly has no zeros outside the unit circle depicted centered at the origin. Perhaps these so-called 'critical points' could, however, cluster on one side of the unit circle, as suggested in the sketch? No: all of our suggested critical points lie outside the unit circles centered at the two leftmost zeros of  $f(z)$ ; Sendov's Conjecture, known to be true for polynomials of degree 8, and for high degree, asserts this cannot happen. In fact,  $f'(z)$  has three zeros at the origin, lying within unit distance of all the zeros of  $f(z)$  (the other four zeros are the fourth roots of 17/144, being approx.  $\pm 0.59$  and  $\pm 0.59i$ ).



**Construction notes:** 1959: The Bulgarian mathematician Blagovest Sendov first proposes his conjecture to Nikola Obreschkov.



1967: Due to a misapprehension, the conjecture is publicised as belonging to fellow Bulgarian Ljubomir Iliev by the influential analyst Walter Kurt Hayman; it becomes widely known as Iliev's (or Ilieff's) Conjecture.

1969: Amram Meir and Ambikeshwar Sharma and prove the conjecture for polynomials of degree  $n \leq 5$ .

1991: After 20 years, Johnny E. Brown of Purdue University pushes on to  $n \leq 6$ .

1996: Julius Borcea achieves  $n \leq 7$ ; Johnny E. Brown's independent proof appears a year later.

1999: Brown and his PhD student Guangping Xiang reach  $n \leq 8$ , now a record of over 10 years standing!

2011: Jérôme Dégot proves the conjecture for large degree (depending on  $z_0$ ).

Zeros

Zeros

Critical points?

Critical points?

**Web link:** [www.ams.org/journals/proc/2014-142-04/S0002-9939-2014-11888-0/](http://www.ams.org/journals/proc/2014-142-04/S0002-9939-2014-11888-0/)  
**Further reading:** *Polynomials* by Victor Prasolov, Springer, 2nd printing, 2009.

