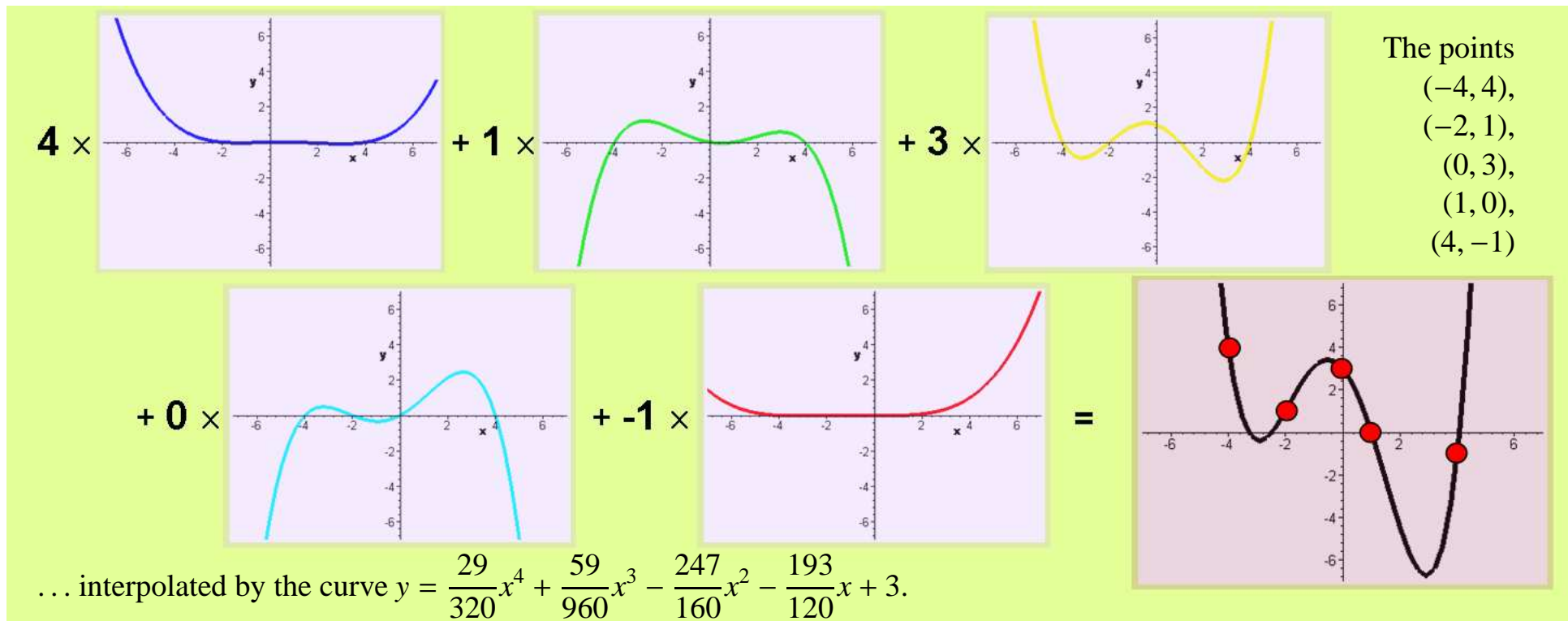# THEOREM OF THE DAY

The Lagrange Interpolation Formula *Given n distinct real values, $x_1, \ldots, x_n$, $n \geq 2$, and any n points, $(x_1, y_1), \ldots, (x_n, y_n)$, in the Cartesian plane, there is unique polynomial curve, $y = p(x)$, of degree $n - 1$, passing these points, specified by*

$$p(x) = \sum_{i=1}^{n} y_i \prod_{i \neq j} (x - x_j)/(x_i - x_j).$$

The points
$(-4, 4)$,
$(-2, 1)$,
$(0, 3)$,
$(1, 0)$,
$(4, -1)$

$4 \times$ $+ 1 \times$ $+ 3 \times$

$+ 0 \times$ $+ -1 \times$ $=$

… interpolated by the curve $y = \dfrac{29}{320}x^4 + \dfrac{59}{960}x^3 - \dfrac{247}{160}x^2 - \dfrac{193}{120}x + 3.$

Remarks:

1. Writing the formula explicitly for $n = 2$ points gives $y = y_1(x - x_2)(x_1 - x_2)^{-1} + y_2(x - x_1)(x_2 - x_1)^{-1}$, the equation for the unique straight line passing through these points: with a little manipulation, it becomes the more memorably symmetrical straight line equation $(y - y_1)/(y_2 - y_1) = (x - x_1)/(x_2 - x_1)$.

2. The $n = 2$ calculation looks very similar to that which solves the Chinese Remainder Theorem for two modular equations, and indeed there is a close connection.

3. The calculation also reveals why the formula works in general: each term is a polynomial which takes the value y_i when $x = x_i$ and is zero when $x = x_j$, $j \neq i$.

4. We can deduce uniqueness thus: suppose $p(x)$ and $q(x)$ are polynomials of degree $n - 1$ through our n points, and define the polynomial $r(x) = p(x) - q(x)$. Now for $i = 1, \ldots, n$, $r(x_i) = 0$ so, by the Factor Theorem, $(x - x_i)$ is a factor of $r(x)$. But this means $r(x)$ has a factor $(x - x_1)(x - x_2) \cdots (x - x_n)$ and must therefore be the zero polynomial, otherwise it has higher degree than p and q which is impossible.

Edward Waring extracted this formula in 1779 from a more general one of Newton. Independently, Lagrange did likewise in 1795.