

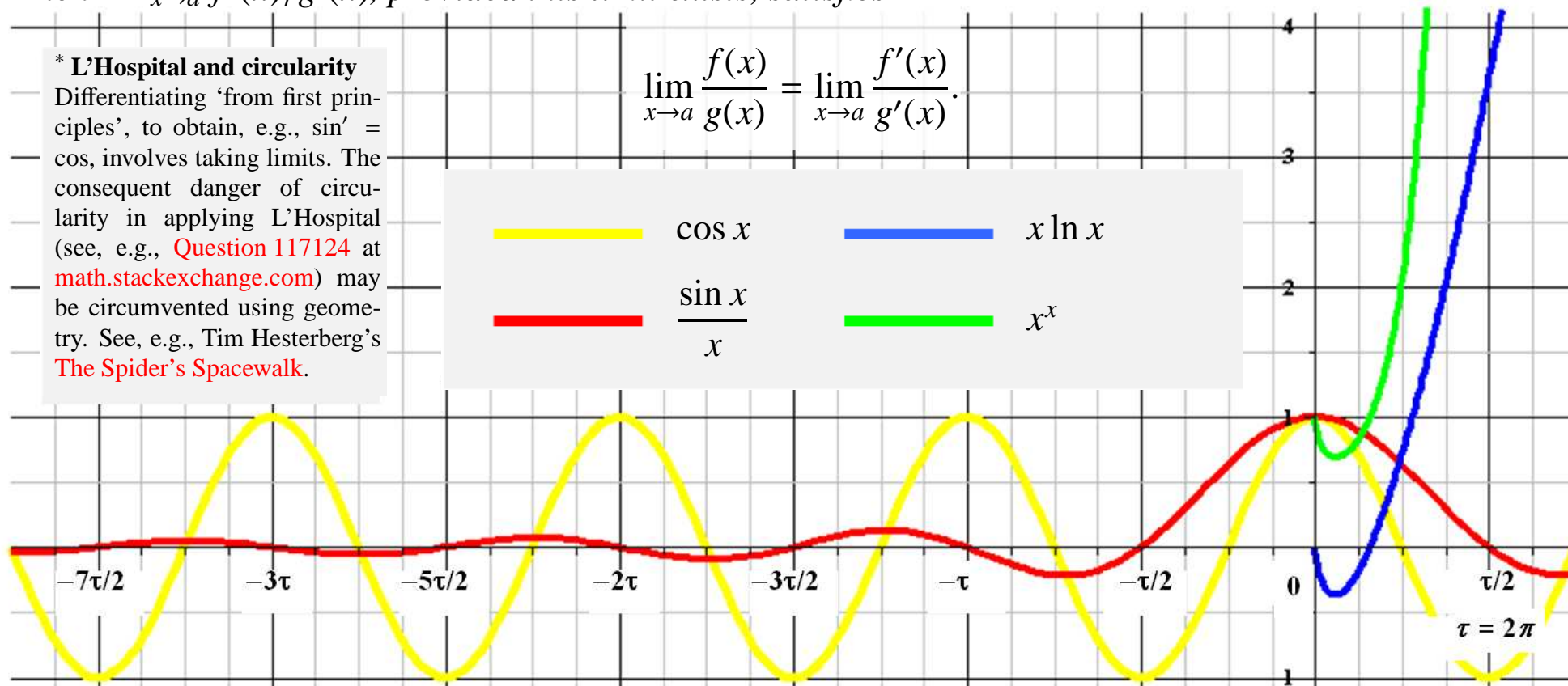
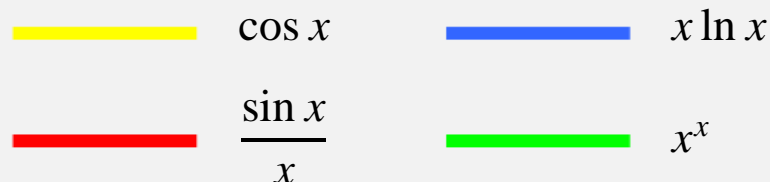


THEOREM OF THE DAY

L'Hospital's Rule Let $f(x)$ and $g(x)$ be real-valued functions of a real variable x . Let $a \in \mathbb{R}$ and suppose that, in some open interval containing a , $f(x)$ and $g(x)$ are differentiable except possibly at $x = a$, with $g'(x)$ being nowhere zero. Suppose, moreover that $\lim_{x \rightarrow a} f = \lim_{x \rightarrow a} g = 0$ or $|\lim_{x \rightarrow a} f| = |\lim_{x \rightarrow a} g| = \infty$. Then $\lim_{x \rightarrow a} f'(x)/g'(x)$, provided this limit exists, satisfies

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

* **L'Hospital and circularity**
Differentiating 'from first principles', to obtain, e.g., $\sin' = \cos$, involves taking limits. The consequent danger of circularity in applying L'Hospital (see, e.g., [Question 117124](http://math.stackexchange.com) at math.stackexchange.com) may be circumvented using geometry. See, e.g., Tim Hesterberg's [The Spider's Spacewalk](#).



It is easy to see that as x approaches 0, the value of $\cos x$ approaches 1, i.e. $\lim_{x \rightarrow 0} \cos x = 1$. Similarly $\lim_{x \rightarrow 0} \sin x = 0$ and $\lim_{x \rightarrow 0} x = 0$, but what about the ratio of these two functions? The '0/0' version of L'Hospital's Rule applies* and gives this limit as $\lim_{x \rightarrow 0} (d \sin x / dx) / (d x / dx) = \lim_{x \rightarrow 0} \cos x / 1 = 1$.

L'Hospital's Rule works equally well with one-sided limits. For example, $x \ln x$ is only defined as we approach zero from the positive direction. We tackle $\lim_{x \rightarrow 0^+} x \ln x$ with a trick: rewrite it as $\lim_{x \rightarrow 0^+} \ln x / (1/x)$. Now $\lim_{x \rightarrow 0^+} \ln x = -\infty$ and $\lim_{x \rightarrow 0^+} 1/x = \infty$, the ' ∞/∞ ' version of L'Hospital's Rule applies and gives $\lim_{x \rightarrow 0^+} (d \ln x / dx) / (d (1/x) / dx) = \lim_{x \rightarrow 0^+} (1/x) / (-1/x^2) = \lim_{x \rightarrow 0^+} -x = 0$.

We have a nice corollary using the fact that $\exp(\lim) = \lim(\exp)$: thus $0^0 = \lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} \exp(\ln(x^x)) = \exp(\lim_{x \rightarrow 0^+} \ln(x^x)) = \exp(\lim_{x \rightarrow 0^+} x \ln x) = e^0 = 1$.

A corollary of the Mean Value Theorem, given by history to Johann Bernoulli (c. 1690); but by posterity to Guillaume de L'Hospital.

Web link: www.math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/ (click on **Math 253**)

Further reading: *A Radical Approach to Real Analysis, 2nd edition* by David M. Bressoud, Mathematical Association of America, 2007, chapter 3.

