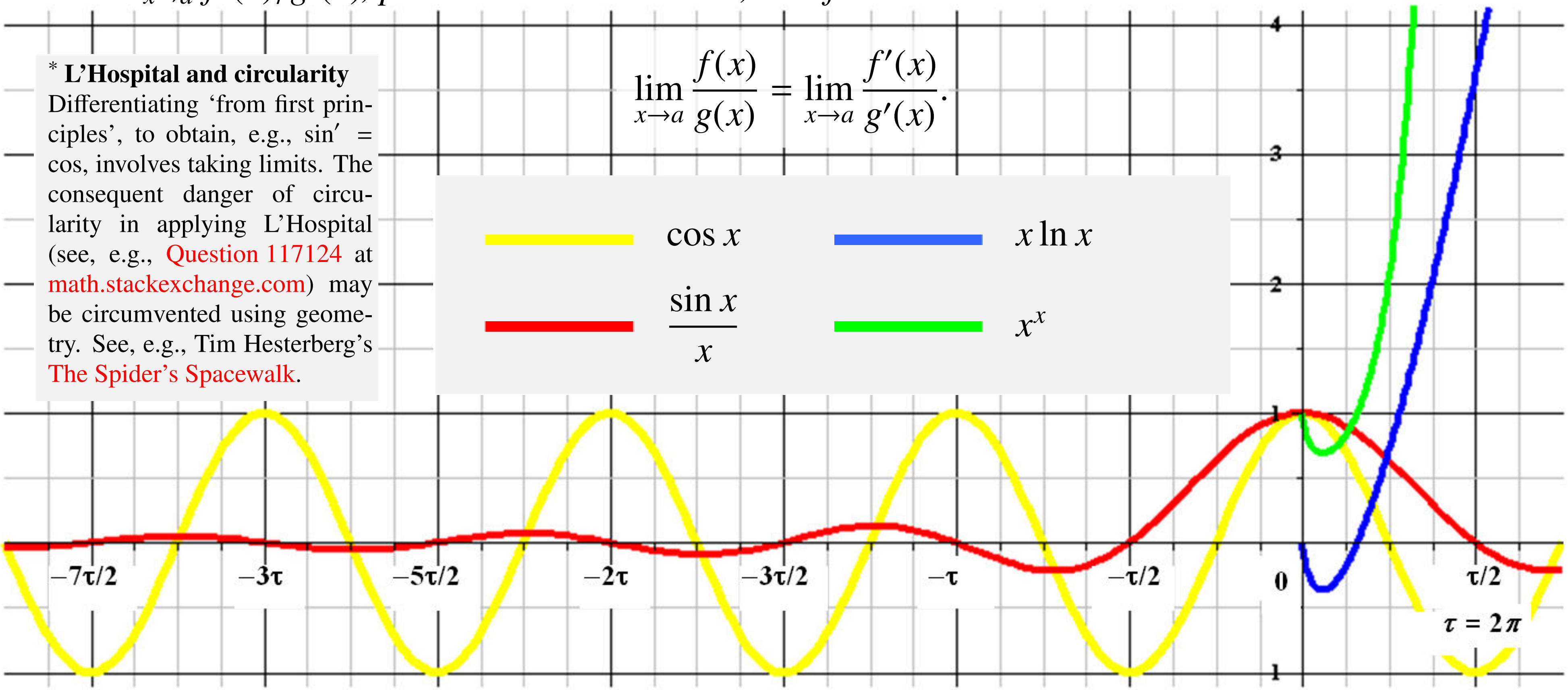
THEOREM OF THE DAY

L'Hospital's Rule *Let $f(x)$ and $g(x)$ be real-valued functions of a real variable x. Let $a \in \mathbb{R}$ and suppose that, in some open interval containing a, $f(x)$ and $g(x)$ are differentiable except possibly at $x = a$, with $g'(x)$ being nowhere zero. Suppose, moreover that $\lim\limits_{x\to a} f = \lim\limits_{x\to a} g = 0$ or $|\lim\limits_{x\to a} f| = |\lim\limits_{x\to a} g| = \infty$. Then $\lim_{x\to a} f'(x)/g'(x)$, provided this limit exists, satisfies*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$



* **L'Hospital and circularity** Differentiating 'from first principles', to obtain, e.g., $\sin' = \cos$, involves taking limits. The consequent danger of circularity in applying L'Hospital (see, e.g., Question 117124 at math.stackexchange.com) may be circumvented using geometry. See, e.g., Tim Hesterberg's The Spider's Spacewalk.

It is easy to see that as x approaches 0, the value of $\cos x$ approaches 1, i.e. $\lim_{x\to 0}\cos x = 1$. Similarly $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0} x = 0$, but what about the ratio of these two functions? The '0/0' version of L'Hospital's Rule applies* and gives this limit as $\lim_{x\to 0}(d\sin x/dx)/(d\,x/dx) = \lim_{x\to 0}\cos x/1 = 1$.

L'Hospital's Rule works equally well with one-sided limits. For example, $x\ln x$ is only defined as we approach zero from the positive direction. We tackle $\lim_{x\to 0^+} x\ln x$ with a trick: rewrite it as $\lim_{x\to 0^+}\ln x/(1/x)$ Now $\lim_{x\to 0^+}\ln x = -\infty$ and $\lim_{x\to 0^+} 1/x = \infty$, the '$\infty/\infty$' version of L'Hospital's Rule applies and gives $\lim_{x\to 0^+}(d\ln x/dx)/(d(1/x)/dx) = \lim_{x\to 0^+}(1/x)\big/(-1/x^2) = \lim_{x\to 0^+} -x = 0$.

We have a nice corollary using the fact that exp(lim) = lim(exp): thus $0^0 = \lim_{x\to 0^+} x^x = \lim_{x\to 0^+}\exp(\ln(x^x)) = \exp(\lim_{x\to 0^+}\ln(x^x)) = \exp(\lim_{x\to 0^+} x\ln x) = e^0 = 1$.

A corollary of the Mean Value Theorem, given by history to Johann Bernoulli (c. 1690); but by posterity to Guillaume de L'Hospital.

Web link: www.math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/ (click on Math 253)

Further reading: *A Radical Approach to Real Analysis, 2nd edition* by David M. Bressoud, Mathematical Association of America, 2007, chapter 3.