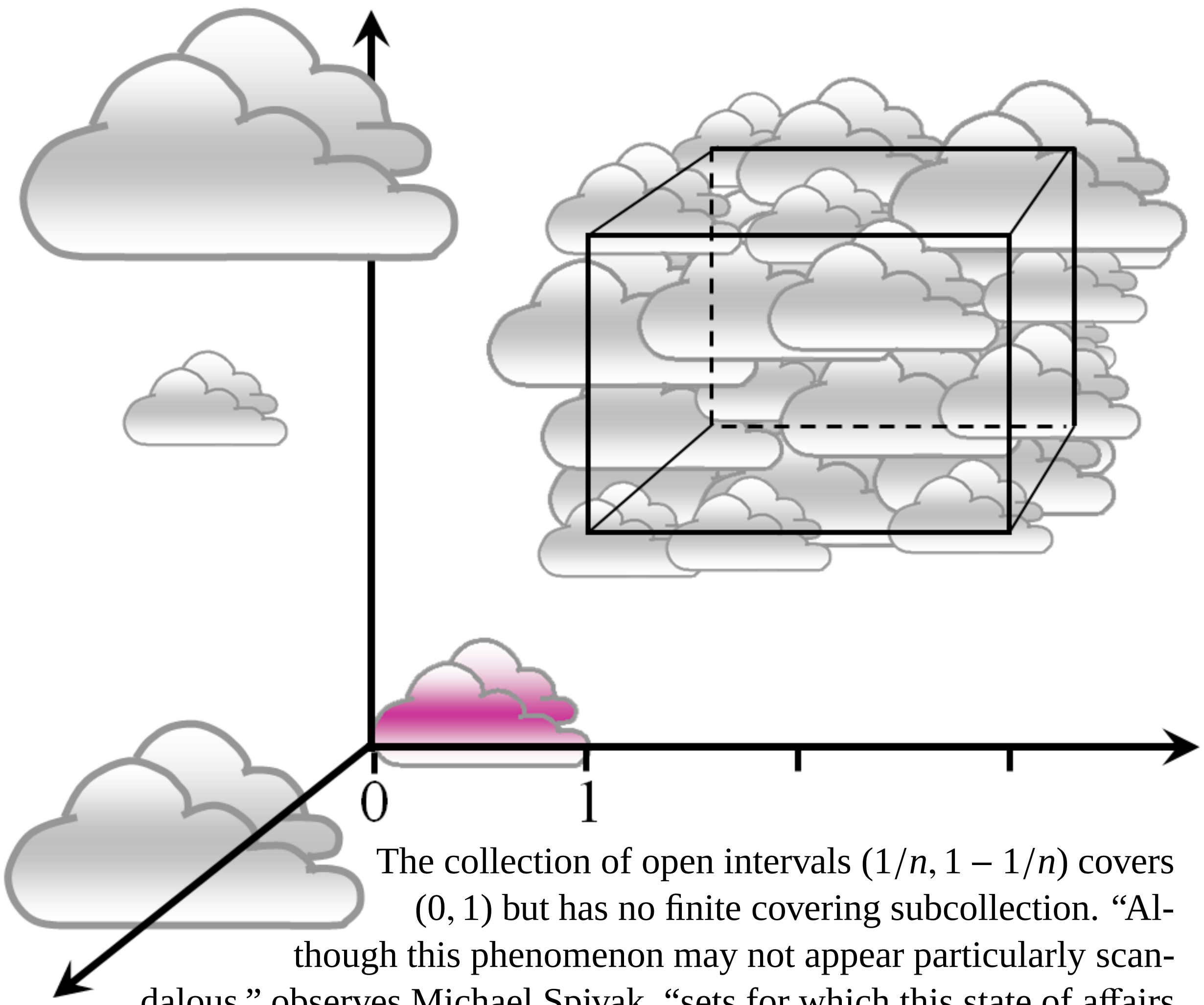
THEOREM OF THE DAY

The Heine–Borel Theorem *The interval* $[0,1]$ *is compact.*



The collection of open intervals $(1/n, 1-1/n)$ covers $(0,1)$ but has no finite covering subcollection. "Although this phenomenon may not appear particularly scandalous," observes Michael Spivak, "sets for which this state of affairs cannot occur are of such importance that they have received a special designation..." Thus the notion of compactness, which goes back to the work of Dirichlet on uniform continuity in the 1850s. Eduard Heine reported Dirichlet's work but appears to deserve only ϵ credit for today's theorem, discovered and proved by Emile Borel in 1895, nearly 15 years after Heine's death, and later referred to by him as "the first fundamental theorem of measure theory".

Compactness is about open sets. Think of an open set as like a cloud: it has no edge, you can never discern the outermost water droplets. On the real line, the open interval $(0,1)$ is defined to include all real numbers from, *but not including*, 0 up to, *but not including*, 1. More generally a subset X of n-dimensional real space, $\mathbb{R}^n$, is **open** if, for any x in X, there is a distance $\epsilon > 0$ small enough so that x can be surrounded by an n-dimensional ball of radius ϵ *which lies entirely within* X. A set is **closed** if its complement is open, and the paradigm of a closed set is $\mathbb{R}\backslash((-\infty,0)\cup(1,\infty)) = [0,1]$: everything between *and including* 0 and 1.

Take a set X in $\mathbb{R}^n$, like the 3-dimensional box shown on the left. Cover it with a, possibly infinite, collection of open sets; this is called an **open cover**. Now X is **compact** if any open cover contains a finite sub-collection of its open sets which still covers X. The Heine–Borel Theorem says $[0,1]$ is compact (whence, by extension, any closed and bounded subset of $\mathbb{R}^n$ is compact). The theorem is essentially equivalent to asserting the **completeness** of the real numbers: any nonempty, bounded subset of $\mathbb{R}$ has a least upper bound. Completeness certainly gives us an easy proof of compactness: suppose C is an open cover of $[0,1]$ and define:

$$A_C = \{x \mid 0 \le x \le 1 \text{ and } [0,x] \text{ has a finite cover in } C\}.$$

Now A_C is nonempty since it contains 0, and it is bounded above by 1 so it has a least upper bound b. Then $b \in [0,1]$ so $b \in U$ for some open set $U \in C$. We now show that: (1) $b \in A_C$, and (2) $b = 1$, whence the theorem. Indeed, U is open so $(b-\epsilon, b+\epsilon) \subseteq U$ for some $\epsilon > 0$; and $b-\epsilon < b$ so $b-\epsilon \in A_C$; so a finite cover C' for $[0, b-\epsilon]$ can be found in C and $C' \cup U$ is a finite cover for $[0,b]$, so $b \in A_C$. Thus (1); now for (2): suppose $b < 1$; then the interval $(b-\epsilon, b+\epsilon) \subseteq U$ contains a point b' with $b < b' < 1$; but now our finite cover $C' \cup U$ is also a finite cover for $[0,b']$, contradicting the fact that b is an upper bound for A_C; so $b = 1$, and we are done.

Web link: arxiv.org/abs/0808.0844

Further reading: *Calculus on Manifolds* by Michael Spivak, Westview Press, 1971 (the proof of Heine–Borel given above is adapted from Chapter 1).