THEOREM OF THE DAY

The Wedderburn-Artin Theorem *Any finite dimensional semisimple algebra is isomorphic to a direct product of matrix algebras over division algebras.*

A

a semisimple algebra

x_{11}	x_{12}
x_{21}	x_{22}

$x_{ij}^2 = x_{ij}$ (idempotent)

$x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$ (commutative)

$x_{ij} \cdot x_{kl} = 0$ if $i = k$ or $j = l$ but not both

E.g. $(x_{11} - 2x_{12}x_{21})(x_{22} - x_{12} - x_{11}x_{22}) = x_{11}x_{22} - 0 - x_{11}x_{22} - 0 + 2x_{12}x_{21} + 0$
$= 2x_{12}x_{21}$

The illustration shows a small example of a semisimple algebra, A, over the complex numbers $\mathbb{C}$, generated by taking sums of products of the indeterminates in the 2×2 grid, weighted by elements in $\mathbb{C}$ and subject to the rules above right. These rules, particularly the last one, which says two indeterminates from the same row or column multiply to zero, seem to take us quite far from ordinary arithmetic. But they limit the dimension to 6, the number of possible nonzero products of indeterminates; and they prevent powers of nonzero elements from becoming zero, which for a commutative algebra is sufficient to define the property of being semisimple. So the Wedderburn-Artin Theorem tells us that the arithmetic can be done entirely in terms of matrix arithmetic. In fact, this being a commutative algebra over $\mathbb{C}$, we get the ultimate reduction, to 1×1 matrices: the arithmetic takes place via componentwise addition and multiplication of vectors in $\mathbb{C}^6$.

How does this work? In algebra A the sum $x_{11} + x_{12} + x_{21} + x_{22} - x_{11}x_{22} - x_{12}x_{21} = 1_A$ acts as an *identity* element, (i.e., $a1_A = 1_Aa = a$, for all $a \in A$); you can confirm, for instance, that $x_{11}x_{22} \times 1_A = x_{11}x_{22} + x_{11}x_{22} - x_{11}x_{22} = x_{11}x_{22}$. We 'decompose the identity' into a sum of 6 orthogonal idempotents: $e_1 = x_{11} - x_{11}x_{22}$, $e_2 = x_{22} - x_{11}x_{22}$, $e_3 = x_{12} - x_{12}x_{21}$, $e_4 = x_{21} - x_{12}x_{21}$, $e_5 = x_{11}x_{22}$, and $e_6 = x_{12}x_{21}$: each one squares to itself ($e_i^2 = e_i$), and the product of any other pair is zero ($e_ie_j = 0$ whenever $i \neq j$). Now the isomorphism in the theorem works via the decomposition $A = Ae_1 \times Ae_2 \times \ldots \times Ae_6$. For example, the product in the illustration above becomes $(e_1 + e_5 - 2e_6) \times (e_2 - e_3 - e_6)$, represented in $\mathbb{C}^6$ as $(1, 0, 0, 0, 1, -2) \times (0, 1, -1, 0, 0, -1) = (0, 0, 0, 0, 0, 2)$, which maps back to $2x_{12}x_{21}$, as required.

Although generally known as 'Wedderburn-Artin', it was this 1907 theorem of Joseph Wedderburn, together with his characterisation of finite dimensional simple algebras as matrix rings, that properly established the theory of associative algebras. Emil Artin was later to extend this description into the domain of ring theory. The study of associative algebras evolved in the late nineteenth century, with prototypes of this famous structure theorem due to Theodor Molien and Elie Cartan (the decomposition of commutative semisimple algebras may be attributed to Weierstrass and Dedekind).

Web link: www.math.stonybrook.edu/~aknapp/books/a2-alg.html. Chapter 2 is authoritative on Wedderburn's contributions.

Further reading: *Finite Dimensional Algebras* by Y. A. Drozd and V. V. Kirichenko, translated and with appendix by V. Dlab, Springer Berlin, 1994.