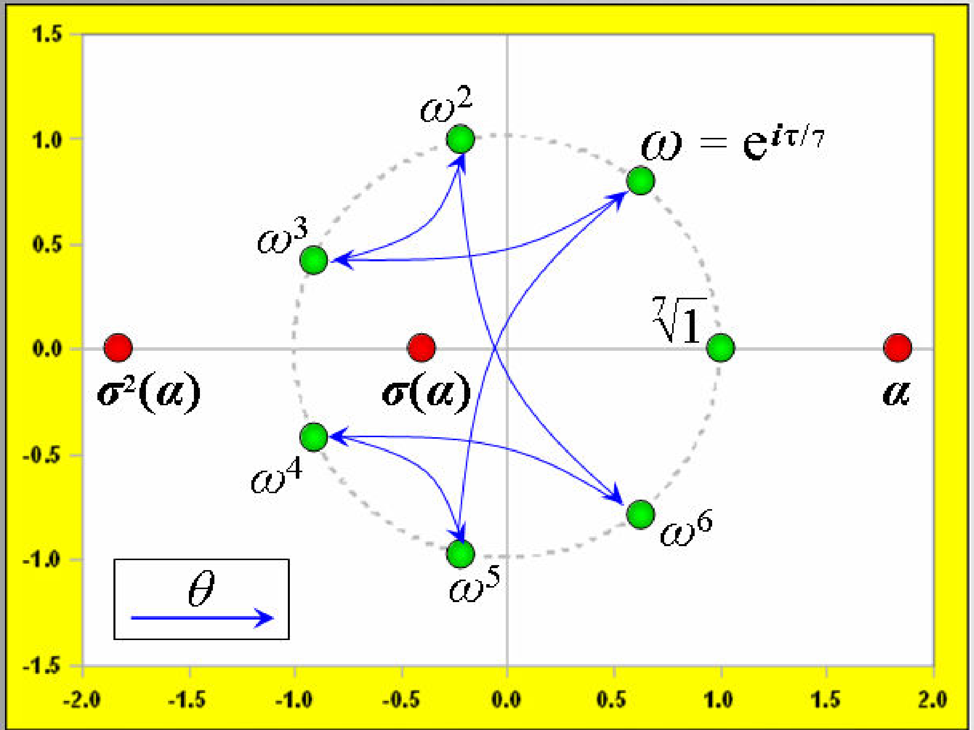
THEOREM OF THE DAY

The Albert-Brauer-Hasse-Noether Main Theorem *Every finite-dimensional noncommutative division algebra over a number field is a cyclic algebra.*

Leonard Dickson's cyclic algebra of degree three, 1914



Base field: $\mathbb{Q}$

Extension field: $L = \mathbb{Q}(\omega) \subset \mathbb{C}$

$\theta : \omega \mapsto \omega^3$ on $\mathbb{Q}(\omega)$

Galois group: $\mathrm{Gal}(L/\mathbb{Q}) = \langle \theta \rangle$

$\alpha = \omega + \omega^6 = 2\cos\tau/7 \quad (\tau = 2\pi)$

Intermediate field: $K = \mathbb{Q}(\alpha) \subset \mathbb{R}$

$\sigma = \theta^2$ on $\mathbb{Q}(\alpha)$

Galois group: $\mathrm{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle$

$\sigma(\alpha) = 2\cos 2\tau/7 = \alpha^2 - 2$

$\sigma^2(\alpha) = 2\cos 3\tau/7 = 1 - \alpha - \alpha^2$

INGREDIENTS

A number field is a finite extension of the rationals $\mathbb{Q}$; cyclic algebras are constructed on *cyclic* field extensions. For example, the equation $x^7 - 1 = 0$ has solutions, such as $\omega = e^{i\tau/7}$, not in $\mathbb{Q}$. The extension $\mathbb{Q}(\omega)$ is the smallest field containing $\mathbb{Q}$ and ω; it is *cyclic of degree 6* because those of its symmetries (permutations preserving the field structure) which leave unmoved all of $\mathbb{Q}$ form a cyclic group of order 6 (namely $C_6 = \{1, \theta, \theta^2, \ldots, \theta^5\}$, $\theta : \omega \mapsto \omega^3$). In 1914, L.E. Dickson, the inventor of cyclic algebras, observed that the square of θ, acting on $\alpha = \omega + \omega^6$, also creates a cyclic group: with $\sigma = \theta^2$, $\{1, \sigma, \sigma^2\} = C_3$. Now α is an irrational number solving $x^3 + x^2 - 2x - 1 = 0$, so we get an *intermediate* extension $\mathbb{Q}(\alpha)$ which is also cyclic. Dickson's cyclic algebra construction adds an indeterminate u, together with the rules: $u^3 = q$, $q \in \mathbb{Q}$ and $u\alpha = (\alpha^2 - 2)u$. This construction generalises to cyclic extensions of any degree over any field; for appropriate choices of q the result is also a noncommutative *division algebra*: a vector space (necessarily having dimension a square) with a multiplication which never gives zero for pairs of nonzero elements. Here, the result is a 9-dimensional algebra over $\mathbb{Q}$, in which a typical element is $a_0 + a_1\alpha + a_2\alpha^2 + a_3u + a_4\alpha u + a_5\alpha^2 u + a_6u^2 + a_7\alpha u^2 + a_8\alpha^2u^2$, $a_i \in \mathbb{Q}$.

Richard Brauer, Helmut Hasse and Emmy Noether announced this result in 1931 with the American A. Adrian Albert close behind. Some of the key ideas were anticipated by Emil Artin's student Käte Hey in her 1927 PhD thesis.

Web link: www.maths.tcd.ie/pub/ims/bull57: click on the excellent survey article by David W. Lewis.

Further reading: *The Brauer-Hasse-Noether Theorem in Historical Perspective* by Peter Roquette, Springer 2004. Dickson's degree-3 cyclic algebra is described in chapter 5 of *A First Course in Noncommutative Rings* by Tsit-Yuen Lam, Springer, 2009.