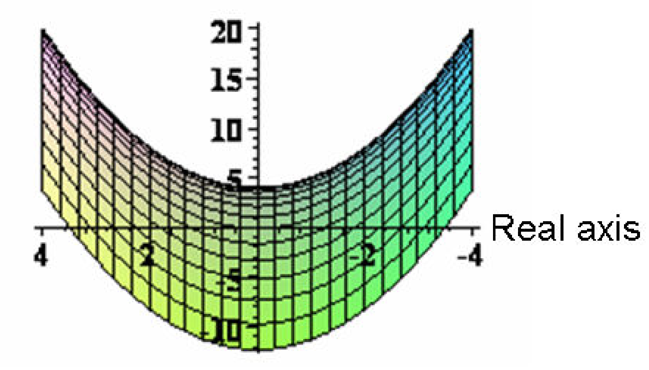
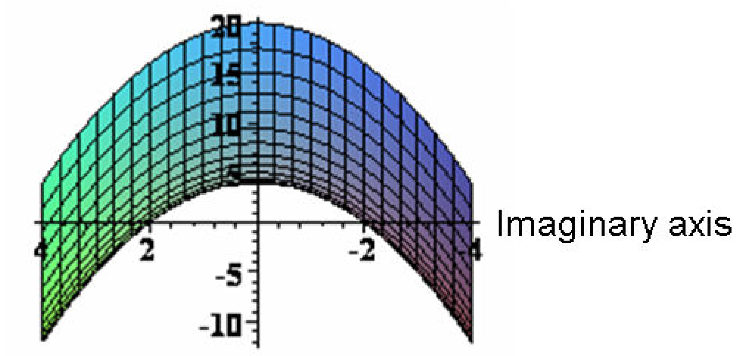
The Fundamental Theorem of Algebra *The polynomial equation of degree n:*

$$z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0,$$

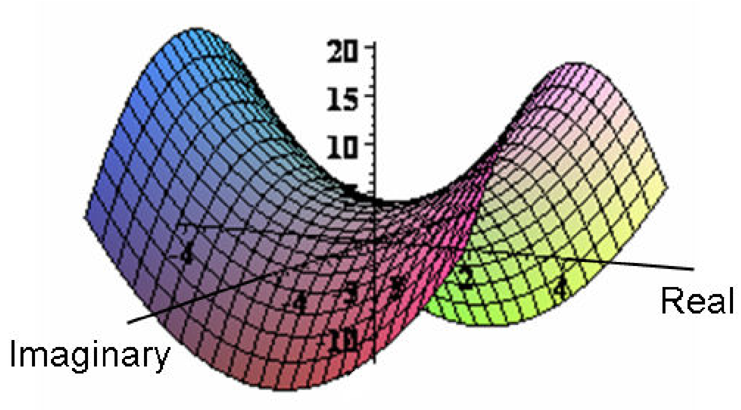
where the a_i belong to $\mathbb{C}$, the complex numbers, has at least one solution in $\mathbb{C}$. As a consequence, the polynomial can be factorised as $(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$, where the α_i are again complex numbers and are precisely the roots of the polynomial.



Real number axis



Purely imaginary number axis



Both axes shown

The graphs depict the real part of the function $f(z) = z^2 + 4$, for z a complex number. When only values of z from the real line are chosen we get the cup-shaped curve described by the upper edge of the surface on the left. The curve never crosses the real (horizontal) axis — it's lowest point is $f(z) = 4$, when $z = 0$ — so there are no real roots. On the other hand, if z is allowed to be a multiple of the *imaginary number* $\sqrt{-1}$, we get the curve described by the *lower* edge of the middle surface, with $z^2 + 4 = 0$ when $z = \pm 2\sqrt{-1}$, since $(\pm 2)^2 \times (\sqrt{-1})^2 = -4$. The complete curve depicting the real part of $z^2 + 4$ for all *complex numbers* of the form $z = x + iy$, where i denotes $\sqrt{-1}$, is shown on the right.

Of course, the analysis of complex functions must also take into account the surface depicting their imaginary part: a root must set both real and imaginary parts to zero! However, the calculations above confirm that we have indeed found two roots; and the fundamental theorem assures us that no others can exist.

The *complex plane* of numbers $x + iy$, depicted using a horizontal axis for the *real part x* and a vertical axis for the *imaginary part y*, is often called the Argand diagram in honour of Jean-Robert Argand (1768–1822) an accountant and amateur mathematician who gave the first full statement and proof of the Fundamental Theorem in 1806 (although a somewhat simpler version had effectively been proved by James Wood in 1798 and by Gauss in his doctoral disseration of 1797).

Web link: William Dunham's George Pólya Award-winning 1992 article at www.maa.org/programs/maa-awards/writing-awards/george-polya-awards.

Further reading: *The Fundamental Theorem of Algebra* by Benjamin Fine and Gerhard Rosenberger, Springer New York, 1997.